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Dihedral homology of commutative algebras

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Abstract

Let A be an associative $\boldsymbol{k}$-algebra with involution, where $\boldsymbol{k}$ is a commutative ring of characteristic not equal to two. Then the dihedral groups act on the Hochschild complex and, following Loday, a new homological theory, called dihedral homology, can be defined generalizing the notion of cyclic homology defined by Connes. Here we give a model to compute dihedral homology of a commutative algebra over a characteristic zero field. As, for an involutive algebra, we have a decomposition of Hochschild homology into a direct sum of two $\boldsymbol{k}$-modules: $\mathbb{Z}_2$-equivariant and skew $\mathbb{Z}_2$-equivariant Hochschild homologies, we give smoothness criteria in terms of vanishing of some $\mathbb{Z}_2$-equivariant Hochschild homology groups.

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0. Introduction

Let $\boldsymbol{k}$ be a unital commutative ring, where 2 is invertible, and let A be an associative $\boldsymbol{k}$-algebra with involution. We recall, [5, 14], that the definition of cyclic homology uses explicitly the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ over the Hochschild complex.

If A is involutive, an action of the dihedral groups $\boldsymbol{D}_n$ on the Hochschild complex can be defined as in [14], and it gives two new homological theories called dihedral homology and skew dihedral homology, see also [6] and [15].

The Hochschild homology of an involutive algebra decomposes into two parts ($\mathbb{Z}_2$-equivariant and skew $\mathbb{Z}_2$-equivariant Hochschild homologies), and the Connes' long exact sequence splits into two long exact sequences relating dihedral homology, skew dihedral homology, $\mathbb{Z}_2$-equivariant and skew $\mathbb{Z}_2$-equivariant Hochschild homologies.

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In this paper we use the techniques of models developed in [3,7] to study $\mathbb{Z}_2$-equivariant Hochschild homology and dihedral homology of a commutative algebra over a characteristic zero field. Most of the results obtained in the above papers can be generalized to dihedral homology. As a consequence, we give a sufficient (and necessary) condition for an involutive graded algebra to be a polynomial algebra, in terms of vanishing of some dihedral homology groups, or $\mathbb{Z}_2$-equivariant Hochschild homology groups. Finally, we show that $\mathbb{Z}_2$-equivariant Hochschild homology behaves well under localization. We use this result to give a positive answer to a refinement of a conjecture by Rodicio [16]. We prove:

Theorem. *Let $A = \mathbb{C}[x_1, \ldots, x_n]/I$ be the coordinate ring of an algebraic variety of the affine space $A_n(\mathbb{C})$, containing the origin and invariant under symmetry by the origin. We endow A with the symmetry induced by $\omega(x_i) = -x_i$, for all i. If A is not smooth at the origin, then there exists an integer $p > 0$ such that $HH_i^+(A) \neq 0$ for all $i < p$ and $HH^+_{p+4n}(A) \neq 0$ for all $n \in N$ (where HH^+ denotes the $\mathbb{Z}_2$-equivariant Hochschild homology).*

1. Dihedral homology of differential graded algebras

Let $\boldsymbol{k}$ be a commutative ring with unit where 2 is invertible. We work in the category $\boldsymbol{k}$-DGA of $\boldsymbol{k}$-differential graded algebras. An object (A, d_A) is the data of a graded module $A = \oplus_{n\in N} A_n$, (A_0 contains $\boldsymbol{k}$), a multiplication on A such that $A_n . A_p$ is included in A_{n+p}, and a derivation d_A of degree -1 satisfying $d_A^2 = 0, d_A(a.b) = (d_A a).b + (-1)^{|a|} a.(d_A b)$, where $|a|$ denotes the degree of $a \in A$.

We assume furthermore that (A, d_A) is endowed with an involution ω, that is a $\boldsymbol{k}$-linear map of degree zero, commuting with d_A, $\omega^2 = Id$, and satisfying

$$\omega(a.b) = (-1)^{|a|.|b|}\, \omega(b).\omega(a).$$

Following [8, 13], we consider the bigraded Hochschild complex $(C_{pq})_{p,q \geq 0}$

$$C_{pq} = \oplus A_{i_0} \otimes \bar{A}_{i_1} \otimes \ldots \otimes \bar{A}_{i_p},$$

where $\bar{A} = A/\boldsymbol{k}$ and the sum ranges over the $(i_0, \ldots, i_p)$ such that $i_0 + \cdots + i_p = q$.

We extend the definition of d_A to this bicomplex, and the definitions of b and B.

Denote $\mathscr{C}_* = \oplus_{n \geq 0} \mathscr{C}_n, \mathscr{C}_n = \oplus_{p+q=n} C_{pq}$.

We recall that $HH_*(A, d_A) = H_*(\mathscr{C}_*, b + d_A)$.

Denote $\mathscr{B}_* = \oplus_{n \in N} \mathscr{B}_n, \mathscr{B}_n = \mathscr{C}_n \oplus \mathscr{C}_{n-2} \oplus, \ldots ,$ and ${}_B b$ its differential

$${}_B b(c_n, c_{n-2}, \ldots) = ((b + d_A)(c_n) + B(c_{n-2}), (b + d_A)(c_{n-2}) + B(c_{n-4}), \ldots ,)$$

We recall that $HC_*(A, d_A) = H_*(\mathscr{B}_*, {}_B b)$

Consider the dihedral group $\boldsymbol{D}_p$ of order $2p$, generated by t and u with the relations $t^p = u^2 = 1$, $u.t.u = t^{-1}$.

If (A, d_A) is endowed with an involution ω, then $\boldsymbol{D}_{p+1}$ acts on C_{pq} by t and u as

$$t(a_0 \otimes \cdots \otimes a_p) = (-1)^{p + |a_p|(|a_{p-1}| + \cdots + |a_1|)}(a_p \otimes a_0 \otimes \cdots \otimes a_{p-1}),$$
$$u(a_0 \otimes \cdots \otimes a_p) = (-1)^{p(p+1)/2 + \Phi(p)}(\omega(a_0) \otimes \omega(a_p) \otimes \cdots \otimes \omega(a_1)),$$

where

$$\Phi(p) = |a_1| \sum_{i>1} |a_i| + |a_2| \sum_{i>2} |a_i| + \cdots + |a_{p-1}|\,|a_p|.$$

Lemma 1.1. *$bu = ub$, $Bu + uB = 0$, $ud_A = d_A u$.*

Proof. We check the two first relations as in [14], the third one is an easy calculation.

So, as in [14], we have a decomposition $\mathscr{C}_* = \mathscr{C}_*^+ \oplus \mathscr{C}_*^-$ into two subcomplexes where u acts as the identity on $\mathscr{C}_*^+$ and acts as $-Id$ on $\mathscr{C}_*^-$.

Let $HH_*^+(A, d_A) = H_*(\mathscr{C}_*^+, b + d_A)$ and $HH_*^-(A, d_A) = H_*(\mathscr{C}_*^-, b + d_A)$, we have

$$HH_*(A, d_A) = HH_*^+(A, d_A) \oplus HH_*^-(A, d_A).$$

HH_*^+ is called *$\mathbb{Z}_2$-equivariant Hochschild homology* and HH_*^- is called *skew $\mathbb{Z}_2$-equivariant Hochschild homology.*

Now, we put

$$\mathscr{B}_n^+ = \mathscr{C}_n^+ \oplus \mathscr{C}_{n-2}^- \oplus \mathscr{C}_{n-4}^+ \oplus \cdots \quad \text{and} \quad \mathscr{B}_n^- = \mathscr{C}_n^- \oplus \mathscr{C}_{n-2}^+ \oplus \mathscr{C}_{n-4}^- \oplus \cdots.$$

Definition 1.2. $H_*(\mathscr{B}_*^+, {}_Bb)$ is called the *dihedral homology* of (A, d_A) and denoted $HD_*(A, d_A)$.

$H_*(\mathscr{B}_*^-, {}_Bb)$ is called the *skew dihedral homology* of (A, d_A) and denoted $HSD_*(A, d_A)$

We have $HC_*(A, d_A) = HD_*(A, d_A) \oplus HSD_*(A, d_A)$

As in [14], there are long exact sequences

$$\cdots \to HH_n^+(A, d_A) \to HD_n(A, d_A) \to HSD_{n-2}(A, d_A) \to HH_{n-1}^+(A, d_A) \to \cdots$$

$$\cdots \to HH_n^-(A, d_A) \to HSD_n(A, d_A) \to HD_{n-2}(A, d_A) \to HH_{n-1}^-(A, d_A) \to \cdots$$

In the rest of the paper, we will work with commutative differential graded algebras. Such an algebra satisfies $a_n . a_m = (-1)^{mn} a_m . a_n$, for $a_n \in A_n, a_m \in A_m$. So, an involutive commutative differential graded algebra has an involution ω which is a morphism in the category of commutative differential graded algebras.

2. Models for $\mathbb{Z}_2$-equivariant Hochschild homology and dihedral homology

Let (A, d_A) be a commutative differential graded algebra endowed with an involution ω. Theorem 1.3 of [10], stated for cochain algebras, remains valid since it relies on the fact that any $\mathbb{Z}/2\mathbb{Z}$-invariant subspace of a vector space has a $\mathbb{Z}/2\mathbb{Z}$-invariant complement. So the construction of Proposition 1.1 of [3] can be performed equivariantly and we have the following.

Theorem 2.1. *Let (A, d_A) be a commutative differential graded algebra endowed with an involution ω. Then there exists a free commutative differential graded algebra $(\Lambda V, \partial)$ and a morphism $\rho : (\Lambda V, \partial) \to (A, d_A)$ inducing an isomorphism in homology such that*

(1) $V = \bigoplus_{n \in N} V_n$, on each V_n, there exists an involution ω, which induces a morphism of commutative differential graded algebras,

(2) $\rho\omega = \omega\rho$.

Such an algebra $(\Lambda V, \partial)$ is called an *equivariant model* of (A, d_A).

Remark. Let A be an involutive commutative algebra of finite type, then A is isomorphic to $\boldsymbol{k}[x_1, \ldots, x_p]/I$, where the involution ω of A is the image of ω' on $\boldsymbol{k}[x_1, \ldots, x_p]$ satisfying $\omega'(x_i) = \pm x_i$ for all i, and I contains $\omega'(I)$. So we can construct an equivariant model of A, $(\Lambda V, \partial)$, with $V_0 = \bigoplus_{1 \le i \le p} \boldsymbol{k}x_i$ and $\dim V_n < \infty$, for all n.

Proposition III. 2.9 of [8] can be transposed in this context:

Proposition 2.2. *Let $f : (A, d_A) \to (B, d_B)$ be an equivariant morphism of involutive commutative differential graded algebras over a field. If f_* is an isomorphism from $H_*(A, d_A)$ to $H_*(B, d_B)$, then f induces isomorphisms between $\mathbb{Z}_2$-equivariant (resp. skew $\mathbb{Z}_2$-equivariant) Hochschild homology and dihedral homology.*

From now on, we will assume that $\boldsymbol{k}$ is a field of characteristic zero, and using Proposition 2.2, we will work with the equivariant model $(\Lambda V, \partial)$.

In the appendix of [12], we define the module of differential forms Ω^1 of a commutative graded algebra (A, ∂), extending the classical definition, so that Ω^1 is an (A, ∂)-differential module with a differential δ satisfying $d\partial + \delta d = 0$.

If (A, ∂) is endowed with an involution ω, we define an involution still denoted ω on Ω^1 satisfying $\omega d + d\omega = 0$, $\omega\delta = \delta\omega$.

By definition, $(\Omega^*_{(A,\partial)}, \delta)$ is the (A, ∂)-commutative differential graded algebra on Ω^1. So the formula:

$$\omega_n(a_0 \wedge da_1 \wedge \cdots \wedge da_n) = (-1)^n \omega(a_0) d\omega(a_1) \wedge \cdots \wedge d\omega(a_n)$$

defines an involution ω on $(\Omega^*_{(A,\partial)}, \delta)$ which is a morphism of commutative differential graded algebras satisfying $\omega d + d\omega = 0$.

If $(A, \partial) = (\Lambda V, \partial)$, the algebra $(\Omega^*_{(\Lambda V,\partial)}$, of differential forms has the form $(\Lambda V \otimes \Lambda \bar{V}, \delta)$ with $\bar{V} = dV$, and $\delta d + d\partial = 0$.

Now, we recall the main result of [3] (Theorem 2.4).

Proposition 2.3 (Burghelea and Vigué-Poirrier [3]). *The map*

$$\theta_{p,n-p} : C_{p,n-p}(\Lambda V, \partial) \to (\Omega^p_{(\Lambda V,\partial)})_n$$

defined by

$$\theta_p(a_0 \otimes \cdots \otimes a_p) = [(-1)^{\varepsilon_p(a)}]/p! \cdot (a_0 \wedge da_1 \wedge \cdots \wedge da_p),$$

where $a_0 \in \Lambda V, a_i \in \Lambda V/k$ *if* $i \geq 1$, $\varepsilon_p(a) = |a_1| + |a_3| + \cdots$ *satisfies*

(1) $\theta_0 b = 0, \theta_0 \partial = \delta_0 \theta,\ \theta_0 B = d_0 \theta$;

(2) θ *induces isomorphisms*: $HH_n(\Lambda V, \partial) \cong H_n(\Omega^*, \delta)$ *for all* $n \geq 0$ *and* $HC_n(\Lambda V, \partial) \cong HC_n(\Omega^*, \partial, \delta)$, *where* $HC_*(\Omega^*, \partial, \delta)$ *is the total homology of the bicomplex*

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \cdots & \longleftarrow & (\Omega^*)_n & \xleftarrow{d} & (\Omega^*)_{n-1} & \longleftarrow & \cdots \\ & & \delta\downarrow & & \downarrow\delta & & \\ \cdots & \longleftarrow & (\Omega^*)_{n-1} & \xleftarrow{d} & (\Omega^*)_{n-2} & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \end{array}$$

Lemma 2.4. *The following diagram commutes*

$$\begin{array}{ccc} C_p(\Lambda V, \partial) & \xrightarrow{\theta_p} & \Omega^p_{(\Lambda V, \partial)} \\ u\downarrow & & \downarrow\omega_p \\ C_p(\Lambda V, \partial) & \xrightarrow[\theta_p]{} & \Omega^p_{(\Lambda V, \partial)} \end{array}$$

Proof. Left to the reader.

We have a decomposition $\Omega^*_{(\Lambda V, \partial)} = (\Omega^*)^+ \oplus (\Omega^*)^-$ where $(\Omega^*)^+ = \{x/\omega(x) = x\}$ and $(\Omega^*)^- = \{x/\omega(x) = -x\}$.

From Proposition 2.3 and Lemma 2.4, we have directly:

Theorem 2.5. *We have explicit isomorphisms, induced by* θ, *for each* $n \geq 0$.

$$HH_n^+(A) \cong HH_n^+(\Lambda V, \partial) \cong H_n((\Omega^*)^+, \delta) = \textstyle\bigoplus_i H_n^{(i)}((\Omega^*)^+, \delta)$$

$$HD_n(A) \cong HD_n(\Lambda V, \partial) \cong HC_n((\Omega^*)^+, \delta, d) = \textstyle\bigoplus_i HC_n^{(i)}((\Omega^*)^+, \delta, d),$$

where

$H_*^{(i)}((\Omega^*)^+, \delta) = H_*((\Omega^*)^+ \cap (\Omega^i, \delta))$

$HC_*^{(i)}((\Omega^*)^+, \delta, d)$ *is the total homology of the bicomplex*

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \cdots & \longleftarrow & (\Omega^i_n)^+ & \xleftarrow{d} & (\Omega^{i-1}_n)^- & \longleftarrow & \cdots \\ & & \delta\downarrow & & \downarrow\delta & & \\ \cdots & \longleftarrow & (\Omega^i_{n-1})^+ & \xleftarrow{d} & (\Omega^{i-1}_{n-1})^- & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \end{array}$$

Since $H_n(\Omega_*^+, d) = ((\Omega_n^+ \cap \mathrm{Ker}\, d)/d(\Omega_n^-)$ and

$$H_n(\Omega_*^-, d) = ((\Omega_n^- \cap \mathrm{Ker}\, d)/d((\Omega_n^+) \quad \text{for all } n > 0,$$

we have a similar result to Theorem 2.1 of [3]

Theorem 2.6. *The map* $\phi : \Omega_n^+ \oplus \Omega_{n-2}^- \oplus \cdots \to (\Omega_{n+1}^- \cap d(\Omega_n^+), \delta)$ *defined by* $\phi(c_n, c_{n-2} \ldots) = (-1)^n dc_n$ for $c_{n-2i} \in \Omega_{n-2i}$, *is a morphism of complexes and induces an isomorphism between* $H\bar{D}_*(\Lambda V, \partial) = HD_*(\Lambda V, \partial)/HD_*(\boldsymbol{k})$ *and* $H_{*+1}(\Omega_*^- \cap d(\Omega_*^+), \delta)$. *Analogously, we have an isomorphism between* $HS\bar{D}_*(\Lambda V, \partial) = HSD_*(\Lambda V, \partial)/HSD_*(\boldsymbol{k})$ *and* $H_{*+1}(\Omega_*^+ \cap d(\Omega_*^-), \delta)$.

The famous Hochschild–Kostant–Rosenberg theorem implies that if A is smooth, then the $\mathbb{Z}_2$-equivariant Hochschild homology groups $HH_n^+(A)$ are zero for n sufficiently large.

For graded algebras, we can prove a converse of this result, using the theory developed in the present paragraph. This is, in fact, the proof of a refinement of a conjecture by Rodicio [16].

Theorem 2.7. *Let A be a graded algebra over a characteristic zero field, and ω an involution on A. If there exists three integers i, j, k such that $i - j$, $j - k$ and $i - k$ are not divisible by* 4, *and*

$$HH_i^+(A) = HH_j^+(A) = HH_k^+(A) = 0$$

then A is a polynomial algebra.

Proof. The proof relies on Theorem 2.6 and the existence of a minimal model for a graded algebra. Then, we proceed as in the proofs of theorems 1 and 2 of [18]. If A is not a polynomial algebra, we write $A = \boldsymbol{k}[x_1, \ldots, x_m]/I$, with $I \neq 0$, and we consider the elements $Z_{m+2n} = (dx_1 \ldots dx_m)(dy)^n$, and their images by the involution ω. Since A is graded, we have short exact sequences:

$$0 \to HS\bar{D}_{n-1}(A, d_A) \to HH_n^+(A, d_A) \to H\bar{D}_n(A, d_A) \to 0$$

$$0 \to H\bar{D}_{n-1}(A, d_A) \to HH_n^-(A, d_A) \to HS\bar{D}_n(A, d_A) \to 0$$

The elements Z_{m+2n} define nonzero classes in $H\bar{D}_{m+2n-1}$ or $HS\bar{D}_{m+2n-1}$, depending on the actions of ω. This allows us to determine when the groups $HH_n^+(A)$ are not zero.

Remark. In [18], it is proven that if A is not a polynomial algebra, then $HC_n(A) \neq 0$ for infinitely many n. Here we cannot prove the same result for dihedral homology or skew dihedral homology, but instead, it is valid for $\mathbb{Z}_2$-equivariant Hochschild homology.

3. Localization of $\mathbb{Z}_2$-equivariant Hochschild homology. Applications

Let A be a commutative algebra. One of the most important properties of Hochschild homology, specially for geometrical applications, is that it is well-behaved with respect to localization. Explicitly, if S is a multiplicatively closed subset of A, and $A_S = S^{-1}A$, then by a result of Brylinski [2]

$$HH_*(A_S) = HH_*(A) \otimes_A A_S$$

If A is provided with an involution ω, and A^+ is the subalgebra of the elements of A fixed by ω, then $HH_n^+(A)$ is no more an A-algebra but an A^+-algebra.

Let S be a multiplicatively closed subset of A, stable by the involution (i.e. $\omega(S)$ is included in S), and let $S^+ = \{s \in S / \omega(s) = s\}$.

Then $1 \in S^+$, and S^+ is also a multiplicatively closed subset of A.

If $a, a' \in A$, $s, s' \in S$, then $a/s = a'/s'$ in A_S if and only if $\exists t \in S$ such that $t.(as' - a's') = 0$. In this case, $\omega(a)/\omega(s) = \omega(a')/\omega(s')$ in A_S.

So, the formula $\omega(a/s) = \omega(a)/\omega(s)$ makes sense and defines an involution on A_S.

Lemma 3.1. *The inclusion $i: A_{S^+} \to A_S$; $i(a/s) = a/s$ is an isomorphism of algebras, such that $\omega i = i\omega$.*

Proof. It is clear that i is a morphism of algebras which is injective.
It is also surjective because if $a/s \in A_S$, then $a/s = a.\omega(s)/s.\omega(s)$ in A_S, and $s.\omega(s) \in S^+$.

As a consequence of this lemma, from now on we can suppose $S = S^+$.

Consider now an A-bimodule M, which is A^+-symmetric (i.e. $rm = mr$, for $r \in A^+, m \in M$), provided with an involution ω_M compatible with ω.

More explicitly, ω_M is k-linear, $\omega_M^2 = id_M$, and if $a, b \in A, m \in M$, then $\omega_M(a.m.b) = \omega(b) \cdot \omega_M(m) \cdot \omega(a)$. We denote by $M^+ = \{m \in M / \omega_M(m) = m\}$.

As in the previous sections, the Hochschild complex $C_*(A, M)$ can be decomposed into $C_*^+(A, M)$ and $C_*^-(A, M)$, whose homologies are, respectively, $H_*^+(A, M)$ and $H_*^-(A, M)$ [14].

$H_*(A, M)$, (resp. $H_*^+(A, M)$) has a natural structure of symmetric A-bimodule (resp. A^+-bimodule).

If S is a multiplicatively closed subset of A, suppose $S = S^+$, and define $M_S = A_S^+ \otimes_{A^+} M \otimes_{A^+} A_S^+$.

Remark. $(M^+)_S \cong (M_S)^+$ as A_S^+-bimodule.

Theorem 3.2. *In the above conditions,*

$$H_*^+(A_S, M_S) \cong [H_*^+(A, M)]_S \quad \textit{(and analogously for } H^-\textit{)}$$

Proof. First observe that the functor $X \to X^+$ from the category of symmetric A-bimodules to the category of A^+-bimodules is well-defined and exact.

Also, let $\eta_0 : [H_0(A, M)]_S \to H_0(A_S, M_S)$ be the natural isomorphism induced by $\bar{f} : M/[A, M] \to M_S/[A_S, M_S]$; $\bar{f}(\bar{m}) = cl(1 \otimes m \otimes 1)$.

By a theorem of Grothendieck [9], as η_0 is an isomorphism and we also have natural functors $\eta_n : [H_n(A, M)]_S \to H_n(A_S, M_S)$ for $n \geq 0$, then $[H_*(A, M)]_S$ is isomorphic to $H_*(A_S, M_S)$.

Also, η_0 commutes with the involution. So, $[H_*^+(A, M)]_S \cong [(H_*(A, M))^+]_S$. By the previous remark, this last term is identical with $([H_*(A, M)]_S)^+$, and by the result of Brylinski, this equals $[H_*(A_S, M_S)]^+ = H_*^+(A_S, M_S)$.

Now, we apply Theorem 3.2 to the characterization of smoothness in terms of the nullity of some $\mathbb{Z}_2$-equivariant Hochschild homology groups.
In [16], the author conjectures:

Let $\mathbf{k}$ be a field of characteristic zero and let A be a $\mathbf{k}$-algebra of finite type. If $HH_n(A) = 0$ for n sufficiently large, then A is a smooth $\mathbf{k}$-algebra.

In [4, 1], the authors prove the conjecture, under the less restrictive assumption that there exists two Hochschild homology groups HH_{2i} and HH_{2j+1} which vanish.

Here, we give a similar result for involutive commutative algebras.

Theorem 3.3. *Let $A = \mathbb{C}[x_1, \ldots, x_m]/I$ be a reduced commutative algebra of finite type. We assume that A is the coordinate ring of an algebraic subset V containing the origin and symmetric by the origin (so, the involution $\omega(x_i) = -x_i$ for all i, induces an involution on A).*

Then, if V is not smooth at the origin, there exists an integer p such that $HH_i^+(A) \neq 0$ for all $i < p$, and $HH_{p+4n}^+(A) \neq 0$ for all $n \in N$.

Proof. We recall that an algebraic subset V of the affine space $A_m(\mathbb{C})$ is defined by the data of a family of polynomials $(P_i)_{i \in I}$, $P_i \in \mathbb{C}[x_1, \ldots, x_m]$ and

$$V = \{(a_1, \ldots, a_m) \in \mathbb{C}^m / P_i(a_1, \ldots, a_m) = 0, \text{ for all } i\}.$$

If we denote by $I(V)$ the ideal generated by the polynomials Q such that $Q(a_1, \ldots, a_m) = 0$, for all $(a_1, \ldots, a_m) \in V$, then $I(V)$ is equal to the radical of the ideal generated by the family $(P_i)_{i \in I}$. Then $A = \mathbb{C}[x_1, \ldots, x_m]/I(V)$ is called the coordinate ring of V. From the Nullstellensatz theorem, we have a one-to-one correspondence between reduced commutative algebras of finite type and coordinate rings of algebraic subsets.

Now consider an algebraic set V containing the origin O. Let σ be the central symmetry of center O in $A_m(\mathbb{C})$, we assume that V contains $\sigma(V)$. We denote by ω the algebra morphism on $\mathbb{C}[x_1, \ldots, x_m]$ defined by $\omega(x_i) = -x_i$ for all i. If $\sigma(V)$ is a subset of V we can find generators $P_1, \ldots, P_r$ of $I(V)$ such that $\omega(P_j) = \pm P_j$, for all $j \in [1, \ldots, r]$. In the following, $A = \mathbb{C}[x_1, \ldots, x_m]/I(V)$ will be endowed with the image of this involution ω. Let $\mathfrak{M}' = (x_1, \ldots, x_m)$ and $\mathfrak{M} = \mathfrak{M}'/I$. From Theorem 3.2, we have $HH_*^+(A_{\mathfrak{M}}) \cong HH_*^+(A) \otimes_{A^+} (A^+)_{S^+}$, with $S^+ = \{s \in \mathbb{C}[x_i] - \mathfrak{M} / \omega(s) = s\}$.

So we work with the local ring $A_{\mathfrak{M}}$ endowed with the induced involution. Since $\omega(x_i) = -x_i$ for all i, the ideal $\mathfrak{M}'$ has a minimal set of generators on which ω operates as -Id. A classical argument [I], shows that we can write $A_{\mathfrak{M}} = A_0/J$ with A_0 a local regular ring of maximal ideal $\mathfrak{N}$, J is contained in $\mathfrak{N}^2$ and A_0 has an involution ω that operates as $-Id$ on a minimal set of generators $(f_1, \ldots, f_p)$ of $\mathfrak{N}$. Furthermore, we have $A_0/\mathfrak{N} \cong \mathbb{C}$.

Tate's construction [17], allows us to say that there exists a minimal commutative graded differential algebra $(A_0 \otimes \Lambda V, \partial)$, $V = \bigoplus_{n \geq 1} V_n$, and a map from $(A_0 \otimes \Lambda V, \partial)$ onto A_0/J which induces an isomorphism in homology.

On the other hand, since A is involutive, we can build this model such that each V_n is endowed with an involution which is a morphism of commutative differential graded algebras, extending the involution of A_0.

In [7, 19], it is proved that the Hochschild homology of A_0/J is isomorphic to the homology of $(\Omega^*_{A_0} \otimes \Omega^*_{\Lambda V}, \delta)$ with $\delta d + d\partial = 0$.

A similar argument to those of Section 2 shows that

$$HH^+_*(A_{\mathfrak{M}}) = H_*((\Omega^*_{A_0} \otimes \Omega^*_{\Lambda V})^+, \delta)$$

Then the proof is the same as in [1]; if $A_{\mathfrak{M}}$ is not local regular, then $J \neq 0$, so we have $V_1 \neq 0$, we can find an element $y \in V_1$ such that $\omega(y) = \pm y$. Since $\omega(f_i) = -f_i$ for all i, we have $\omega(df_i) = df_i$.

For $n \in N$, we put $Z_{4n+p} = (df_1 \ldots df_p)(dy)^{2n}$, then $\omega(Z_{4n+p}) = Z_{4n+p}$, so $Z_{4n+p} \in (\Omega^*_{A_0} \otimes \Omega^*_{\Lambda V})^+$.

We conclude as in [1].

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